THE KING'S SCHOOL

2006
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Attempt Questions 1-8
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\displaystyle\int \frac{1}{x^2 - 8x + 15}\, dx$ **3**

(ii) $\displaystyle\int \frac{1}{\sqrt{x^2 - 8x + 15}}\, dx$ **2**

(b) (i) Show that $\displaystyle\int \frac{1 - x^2}{1 + x^2}\, dx = 2\tan^{-1} x - x + c$ **2**

(ii) Use the substitution $u = \cos x$ to find the exact value of

$$\int_0^{\pi} \frac{\sin^3 x\, dx}{1 + \cos^2 x}$$ **5**

(c) Find $\displaystyle\int \sin 3\theta \cos \frac{\theta}{2}\, d\theta$ **3**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) If $z = \cos\theta + i\sin\theta$ prove that $\dfrac{1}{z} = \bar{z}$ **2**

(b) The complex number w is given by $w = \dfrac{\sqrt{3} - i}{1 + i}$

Find

(i) $|w|$ **2**

(ii) the exact value of $\arg w$ **2**

(iii) w^6 in the form $a + ib$ **2**

(c) If the point P in a complex plane corresponds to the complex number z, find the equation of the locus of P when $|z + 2i| = |z - 1|$. **3**

(d) Sketch the locus of z if $\dfrac{z - 2i}{z - 1}$ is purely imaginary. **4**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

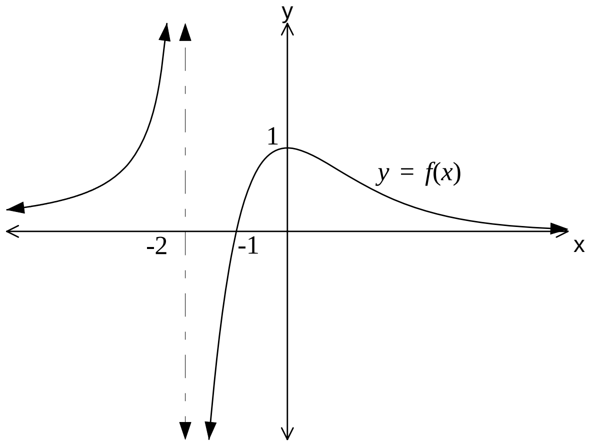
(a) (i) Sketch on the same number plane

$y = |x| - 2$ and

$y = 4 + 3x - x^2$ **2**

(ii) Hence, or otherwise, solve $\dfrac{|x| - 2}{4 + 3x - x^2} > 0$ **2**

(b) The graph of $y = f(x)$ is shown below:



Draw sketches of the following:

(i) $y = [f(x)]^2$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = f'(x)$ **2**

(c) Given that the first and second derivatives of $y = \sin^{-1}\left(x^2 - 1\right)$ are

$$\frac{dy}{dx} = \frac{2x}{\sqrt{2x^2 - x^4}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2x^4}{\left(2x^2 - x^4\right)^{\frac{3}{2}}},$$

draw a neat sketch of $y = \sin^{-1}\left(x^2 - 1\right)$ clearly showing the endpoints, x and y intercepts and concavity. **5**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Sketch the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ showing the coordinates of its foci, its directrices and its asymptotes. **3**

(b) (i) Derive the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P\ (x_1, y_1)$. **3**

(ii) Tangents to the ellipse $x^2 + 4y^2 = 4$ at the points $P\ (2\cos\theta, \sin\theta)$ and $Q\ (2\cos\phi, \sin\phi)$ are at right angles to each other. Show that $4\tan\theta\tan\phi = -1$. **3**

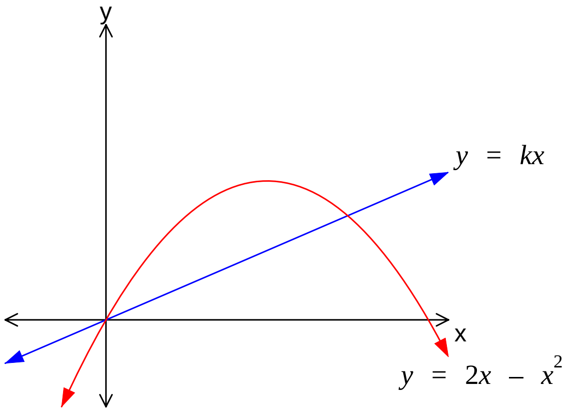
(c) Points $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ on the rectangular hyperbola $xy = c^2$ are such that the tangent at Q passes through the foot N of the perpendicular from P to the *x*-axis. Show that the locus of the midpoint of the chord PQ is the rectangular hyperbola $xy = \frac{9}{8}c^2$. **6**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The area bounded by the curves $y = e^{-x}$ and $y = -e^{-x}$ between $x = 0$ and $x = 1$ is rotated about the line $y = -1$. Find the volume of the solid of revolution formed. **4**

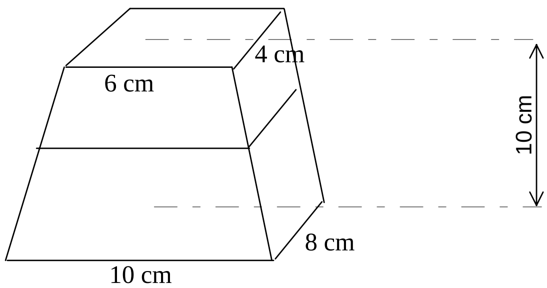
(b) The area bounded by the functions $y = 2x - x^2$ and $y = kx$, $k \geq 0$ is rotated about the y-axis.



Use the method of cylindrical shells to show the volume formed is given by

$V = \dfrac{\pi(2-k)^4}{6}$ units ^{3}. **5**

(c) A solid wooden pedestal of height 10 cm has plane sides. Cross sections parallel to the top and bottom are rectangular. The pedestal measures 4 cm by 6 cm at the top and 8 cm by 10 cm at the bottom.



By using the slicing technique show that the volume of the pedestal is $\frac{1480}{3}$ cm^3. **6**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the polynomial of $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$.

(i) If $P(x)$ has zeros $a + bi, a - 2bi$ (where a, b are real) find the values of a and b. **3**

(ii) Hence factorise $P(x)$ over the field of real numbers. **2**

(b) Given α, β, γ are the roots of $x^3 + 2x^2 - 3x - 4 = 0$.

(i) Evaluate $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) Evaluate $\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\gamma^2} + \dfrac{1}{\gamma^2\alpha^2}$ **2**

(iii) Form an equation whose roots are $\dfrac{2}{\alpha}, \dfrac{2}{\beta}, \dfrac{2}{\gamma}$ **2**

(iv) Form an equation whose roots are $\dfrac{\beta\gamma}{\alpha}, \dfrac{\gamma\alpha}{\beta}, \dfrac{\alpha\beta}{\gamma}$ **4**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The well known *Bullet* trains of Japan can achieve very high speeds. If one of these trains is travelling around a cruve of radius 4000 m at a speed of 160 km/h, and the width of the rails is 1.5 m, how much higher than the inner rail must the outer rail be in order for the lateral pressure on the rails to be avoided? Take $g = 9.8\ ms^{-2}$. **5**

(b) (i) A particle is released so that it falls vertically, under the influence of gravity, in a medium whose retardation varies as the square of the velocity. Show that the terminal velocity of the particle is given by $V = \sqrt{\dfrac{g}{k}}$, where g is acceleration due to gravity and k is a positive constant. **2**

(ii) This same particle is projected vertically in the same medium with an initial velocity u.

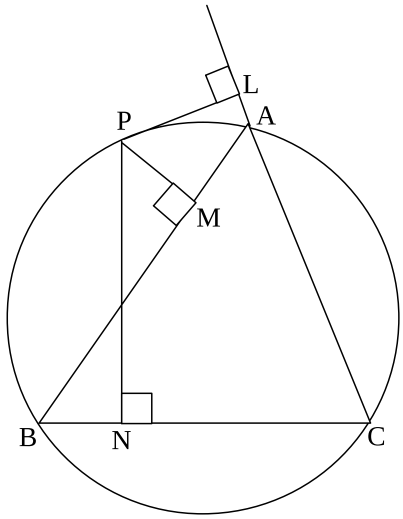
(α) Find an expression for the maximum height reached above the point of projection. **4**

(β) Find an expression for the time taken for the particle to reach the maximum height. **4**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



In the diagram above, ABC is a triangle inscribed in a circle, P is a point on the minor arc AB. L, M and N are the feet of the perpendiculars from P to CA produced, AB and BC respectively.

(i) State a reason why P, M, A and L are concyclic points. **1**

(ii) State a reason why P, B, N and M are concyclic points. **1**

(iii) Show that L, M and N are collinear. **5**

(b) Let $Jn = \int_0^1 x^n e^{-x}\, dx$ for each integer $n \geq 0$.

(i) Show that $J_0 = 1 - \frac{1}{e}$ **1**

(ii) Show that $J_n = nJ_{n-1} - \frac{1}{e}$ **2**

(iii) Prove by mathematical induction that

$$Jn = n! - \frac{n!}{e}\sum_{k=0}^{n}\frac{1}{k!} \text{ for each integer } n \geq 0$$ **3**

(iv) Explain why $Jn \to 0$ as $n \to \infty$ **1**

(v) Deduce that $e = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!}$ **1**

End of Examination

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

EXT 2 MATHS TRIAL HSC 2006

Question 1

(a) (i) $\int \frac{1}{x^2-8x+15}\,dx = \int \frac{1}{(x-5)(x-3)}\,dx$

$$\frac{A}{x-5} + \frac{B}{x-3} \equiv \frac{1}{(x-5)(x-3)}$$

$\therefore A(x-3) + B(x-5) \equiv 1$

Put $x=5$ $\quad \therefore 2A = 1 \quad A = \frac{1}{2}$

$x = 3 \quad -2B = 1 \quad B = -\frac{1}{2}$

$\therefore \int \left(\frac{1}{2x-10} - \frac{1}{2x-6}\right) dx$

$= \frac{1}{2}\ln|2x-10| - \frac{1}{2}\ln|2x-6| + C$

(ii) $\int \frac{1}{\sqrt{x^2-8x+15}}\,dx = \int \frac{1}{\sqrt{(x-4)^2-1}}\,dx$

$= \ln\left(x-4+\sqrt{(x-4)^2-1}\right) + C$

(b)(i) $\int \frac{1-x^2}{1+x^2}\,dx = \int \frac{-1-x^2+2}{1+x^2}\,dx$

$= \int \left(-1 + \frac{2}{1+x^2}\right) dx$

$= 2\tan^{-1} x - x + C$

(ii) $\int_0^{\pi} \frac{\sin^3 x}{1+\cos^2 x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

when $x=0 \quad u=1$

$x=\pi \quad u=-1$

$= -\int_0^{\pi} \frac{\sin^2 x}{1+\cos^2 x} \cdot -\sin x\,dx$

$= -\int_1^{-1} \frac{1-u^2}{1+u^2} \cdot du$

$= \int_{-1}^{1} \frac{1-u^2}{1+u^2}\,du$

$= \left[2\tan^{-1} u - u\right]_{-1}^{1}$

$= (2\tan^{-1} 1 - 1) - (2\tan^{-1}(-1) + 1)$

$= \frac{\pi}{2} - 1 + \frac{\pi}{2} - 1$

$= \pi - 2$

(iii) RTP $J_n = n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}$

Prove true for $n=0$ $\quad J_0 = 0! - \frac{0!}{e} \sum_{k=0}^{0} \frac{1}{0!}$

$= 1 - \frac{1}{e}$

Assume true for n

ie $J_n = n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}$

Prove true for $n+1$

$$J_{n+1} = (n+1) J_n - \frac{1}{e} \quad \text{(from (ii))}$$
$$= (n+1)\left[n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}\right] - \frac{1}{e} \quad \text{(using assumption)}$$
$$= (n+1)! - \frac{(n+1)!}{e} \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{e}$$
$$= (n+1)! - \frac{(n+1)!}{e} \sum_{k=0}^{n} \frac{1}{k!} - \frac{(n+1)!}{e} \cdot \frac{1}{(n+1)!}$$

$\therefore J_{n+1} = (n+1)! - \frac{(n+1)!}{e} \sum_{k=0}^{n+1} \frac{1}{k!}$

$\therefore$ True for $n+1$

Since true for $n=0$ then true for $n=0+1=1$, $1+1=2, \ldots$ $\therefore$ true for all integers $n \geq 0$.

(iv) $\int_0^1 x^n e^{-x}\,dx$ represents the area under $y = x^n e^{-x}$ between $x=0$ and $x=1$.

As $n \to \infty$ $\quad x^n \to 0$ for $0 < x < 1$

$\therefore \int_0^1 x^n e^{-x} = J_n \to 0$.

(v) $\lim_{n\to\infty} J_n = \lim_{n\to\infty} \left(n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}\right)$

$\therefore 0 = \lim_{n\to\infty} \left(n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}\right)$

$\therefore \lim_{n\to\infty} \frac{1}{e} \sum_{k=0}^{n} \frac{1}{k!} = 1$

ie $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = e$

(c) $\int \sin 3\theta \cos \frac{\theta}{2} \, d\theta$.

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$\sin(A-B) = \sin A \cos B - \sin B \cos A$

$\therefore \sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

$\therefore \int \frac{1}{2}\left(\sin\left(3\theta + \frac{\theta}{2}\right) + \sin\left(3\theta - \frac{\theta}{2}\right)\right) d\theta$

$= \frac{1}{2}\int \left(\sin \frac{7\theta}{2} + \sin \frac{5\theta}{2}\right) d\theta$

$= \frac{1}{2} \cdot -\frac{2}{7}\cos\frac{7\theta}{2} + \frac{1}{2} \cdot \left(-\frac{2}{5}\cos\frac{5\theta}{2}\right) + C$

$= -\frac{1}{7}\cos\frac{7\theta}{2} - \frac{1}{5}\cos\frac{5\theta}{2} + C$

Question 2

(a) $z = \cos\theta + i\sin\theta \qquad \therefore |z| = 1$

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}}$$

$$= \frac{\bar{z}}{|z|^2}$$

$$= \bar{z}$$

(b)(i) $w = \dfrac{\sqrt{3} - i}{1 + i}$

$$|w| = \frac{|\sqrt{3} - i|}{|1 + i|}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

(ii) $\arg w = \arg(\sqrt{3} - i) - \arg(1 + i)$

$$= -\frac{\pi}{6} - \frac{\pi}{4}$$

$$= -\frac{5\pi}{12}$$

(iii) $w^6 = (\sqrt{2})^6 \operatorname{cis}\left(6 \times \left(-\frac{5\pi}{12}\right)\right)$

$$= 8 \operatorname{cis}\left(-\frac{5\pi}{2}\right)$$

$$= 8 \operatorname{cis}\left(-\frac{\pi}{2}\right)$$

$$= -8i$$

(c) $|z + 2i| = |z - 1|$

$$\therefore |x + (y+2)i| = |(x-1) + yi|$$

$$\sqrt{x^2 + (y+2)^2} = \sqrt{(x-1)^2 + y^2}$$

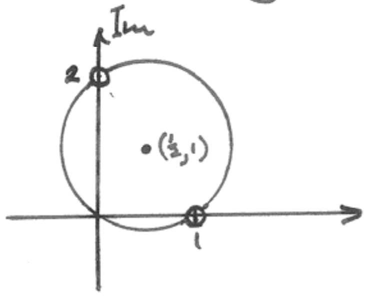
$$x^2 + y^2 + 4y + 4 = x^2 - 2x + 1 + y^2$$

$$\therefore 2x + 4y + 3 = 0$$

(d) $\frac{z-2i}{z-1} = ki$ where $k \neq 0, k \in \mathbb{R}$

$\therefore z - 2i = ki(z-1)$

$\therefore$ The vector representing $z-2i$ is an enlargement or reduction of the anticlockwise ($k>0$) or clockwise ($k<0$) rotation through $\frac{\pi}{2}$ radians of the vector representing $z-1$. $\therefore z$ lies on the circle whose diameter has endpoints $(0,2)$ and $(1,0)$, excluding these endpoints.



circle centre $(\frac{1}{2},1)$
radius $\frac{\sqrt{5}}{2}$
Note: passes through origin

OR ALTERNATIVELY:

$z - 2i = ki(z-1)$

$\therefore x + iy - 2i = ki(x+iy-1)$

$x + (y-2)i = ki((x-1)+iy)$

$\therefore x + (y-2)i = -ky + k(x-1)i$

Equating real & imaginary components

$x = -ky \quad \therefore \quad k = -\frac{x}{y} \quad \ldots ①$

and $y - 2 = k(x-1)$

$\therefore y - 2 = -\frac{x}{y}(x-1) \quad$ (using ①)

$y^2 - 2y = -x^2 + x$

$x^2 - x + \frac{1}{4} + y^2 - 2y + 1 = \frac{5}{4}$

$(x-\frac{1}{2})^2 + (y-1)^2 = \frac{5}{4}$

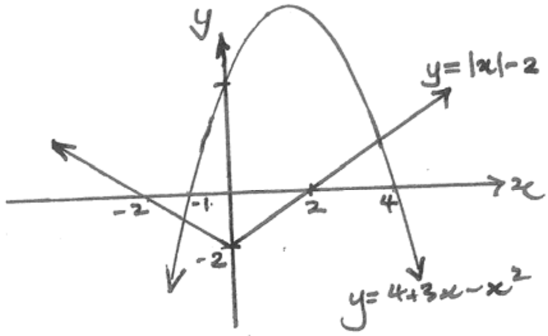
Note $z \neq 2i$ else expression is real and $z \neq 1$ else expression is undefined.
Then sketch as per above!

Question 3

(a) (i) $y = |x| - 2$

$y = 4 + 3x - x^2$

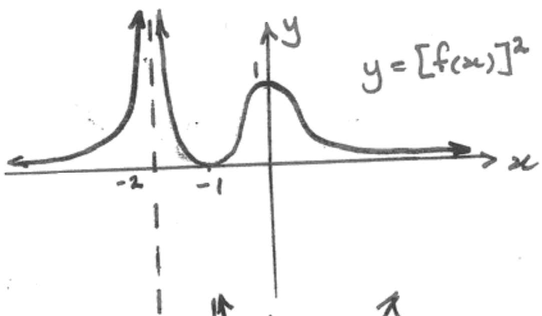
$= (1+x)(4-x)$



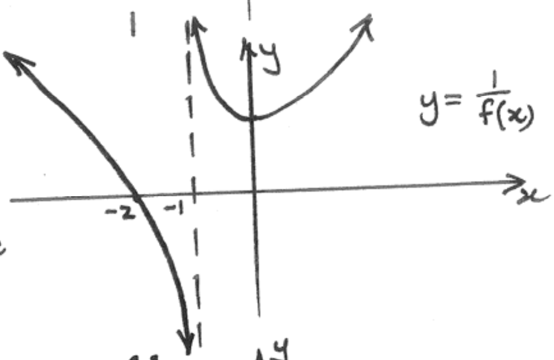
(ii) $\dfrac{|x|-2}{4+3x-x^2} > 0$ when both functions >0 or both <0.

$\therefore -2 < x < -1$, $2 < x < 4$

(b) (i)

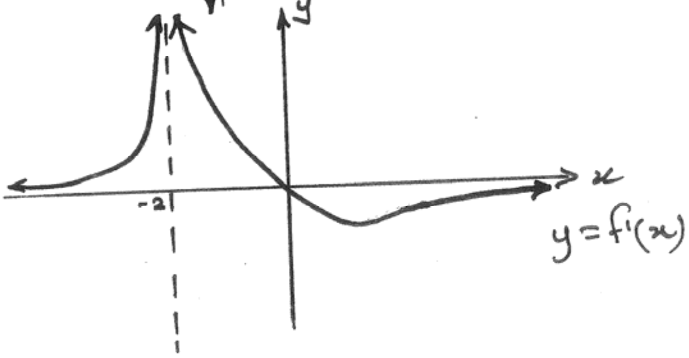


(ii)



(−2,0) may be included or excluded.

(iii)



(c) $\frac{d^2y}{dx^2} > 0 \quad \forall x \neq 0$

and $-1 \leq x^2 - 1 \leq 1$

$0 \leq x^2 \leq 2$

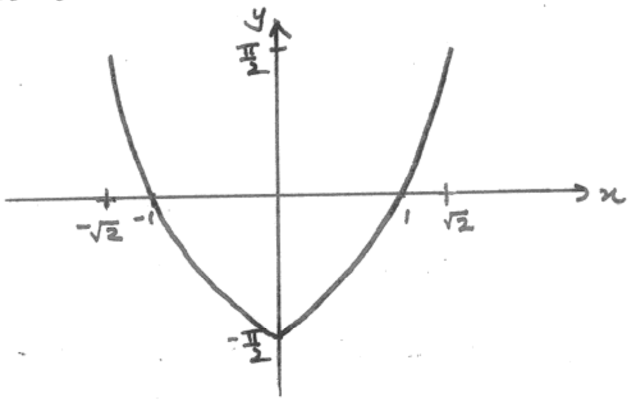
$\therefore -\sqrt{2} \leq x \leq \sqrt{2}$

$\frac{dy}{dx}$ does not exist at $x = 0$

$\frac{dy}{dx} = \frac{2x}{|x|\sqrt{2-x^2}} \quad \therefore \frac{dy}{dx} > 0$ for $x > 0$

& $\frac{dy}{dx} < 0$ for $x < 0$

Also y is an even function.



When $x = \pm\sqrt{2} \quad y = \frac{\pi}{2}$

$x = 0 \quad y = -\frac{\pi}{2}$

when $y = 0 \quad \sin^{-1}(x^2 - 1) = 0$

$\therefore x^2 - 1 = 0$

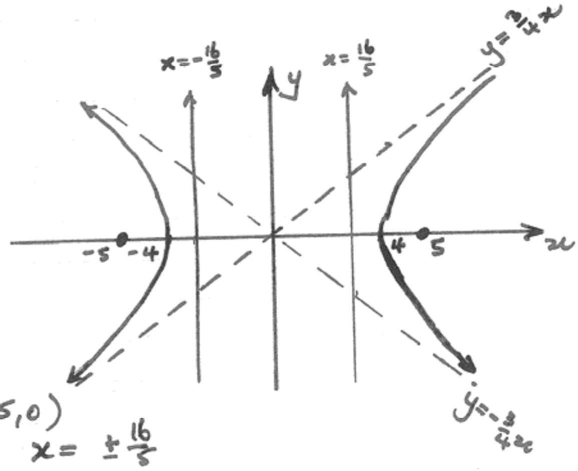
$x = \pm 1$

Note: $(0, -\frac{\pi}{2})$ is not a min T.P.

$\lim_{x \to 0^-} \frac{dy}{dx} = -\sqrt{2}$ and $\lim_{x \to 0^+} \frac{dy}{dx} = \sqrt{2}$.

[redacted]

Question 4



(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a = 4$, $b = 3$

$b^2 = a^2(e^2 - 1)$

$9 = 16(e^2 - 1)$

$\therefore e^2 - 1 = \frac{9}{16}$

$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$

$\therefore$ foci $\pm(ae, 0)$ are $\pm(5, 0)$

directrices $x = \pm\frac{a}{e}$ are $x = \pm\frac{16}{5}$

asymptotes $y = \pm\frac{3}{4}x$

(b) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

$\frac{2y}{b^2}\cdot\frac{dy}{dx} = -\frac{2x}{a^2}$

$\therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

at (x_1, y_1) $\frac{dy}{dx} = -\frac{b^2x_1}{a^2y_1}$

$\therefore y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$

$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{xx_1}{a^2} + \frac{x_1^2}{a^2}$

$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

ie $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(ii) $\frac{x^2}{4} + y^2 = 1$ has tangent $\frac{xx_1}{4} + \frac{yy_1}{1} = 1$

$\therefore$ gradient of tangent is $-\frac{x_1}{4y_1}$

$\therefore$ At P $m_P = -\frac{2\cos\theta}{4\sin\theta}$

$= -\frac{1}{2}\cot\theta$

At Q $m_Q = -\frac{1}{2}\cot\phi$

$\therefore -\frac{1}{2}\cot\theta \cdot (-\frac{1}{2}\cot\phi) = -1$ (since perpendicular)

$\therefore \cot\theta\cot\phi = -4$

$\therefore 4\tan\theta\tan\phi = -1$

(c) $xy = c^2$

$y = \dfrac{c^2}{x}$

$\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

at Q $\dfrac{dy}{dx} = -\dfrac{c^2}{c^2 t_2^2}$

$= -\dfrac{1}{t_2^2}$

Tangent : $y - \dfrac{c}{t_2} = -\dfrac{1}{t_2^2}(x - ct_2)$

$t_2^2 y - ct_2 = -x + ct_2$

$\therefore\ x + t_2^2 y = 2ct_2$

Passes through $(ct_1, 0)$

$\therefore\ ct_1 + 0 = 2ct_2$

$\therefore\ t_2 = \dfrac{t_1}{2}$

Midpoint : $\left(\dfrac{c(t_1+t_2)}{2}, \dfrac{\frac{c}{t_1} + \frac{c}{t_2}}{2}\right)$

$\left(\dfrac{c(t_1+t_2)}{2}, \dfrac{c(t_1+t_2)}{2t_1t_2}\right)$

$\left(\dfrac{c(t_1+\frac{t_1}{2})}{2}, \dfrac{c(t_1+\frac{t_1}{2})}{2t_1 \cdot \frac{t_1}{2}}\right)$

$\left(\dfrac{3ct_1}{4}, \dfrac{\frac{3ct_1}{2}}{t_1^2}\right)$

$\left(\dfrac{3ct_1}{4}, \dfrac{3c}{2t_1}\right)$

$x = \dfrac{3ct_1}{4} \quad \therefore\ t_1 = \dfrac{4x}{3c}$

$y = \dfrac{3c}{2t_1} \quad \therefore\ y = \dfrac{3c}{2\left(\frac{4x}{3c}\right)}$

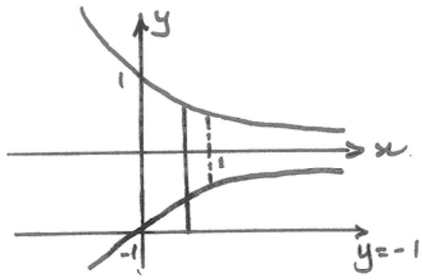
$= \dfrac{3c}{2} \times \dfrac{3c}{4x}$

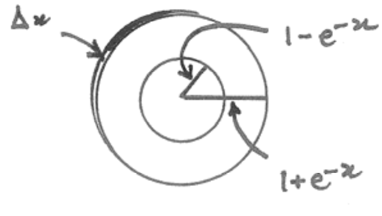
$\therefore\ xy = \dfrac{9}{8}c^2$

Question 5

(a) Area washer:

$$\pi\left((1+e^{-x})^2-(1-e^{-x})^2\right)$$
$$=\pi\left(1+2e^{-x}+e^{-2x}-(1-2e^{-x}+e^{-2x})\right)$$
$$=\pi(4e^{-x})$$
$$=4\pi e^{-x}$$



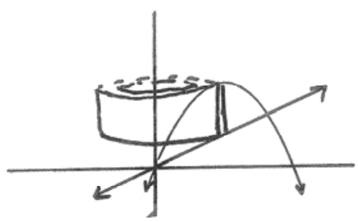


$\therefore$ Volume washer:

$$V=4\pi e^{-x}\,\Delta x$$

$$\therefore V=\int_0^1 4\pi e^{-x}\,dx$$
$$=4\pi\left[-e^{-x}\right]_0^1$$
$$=4\pi\left(-e^{-1}--e^{0}\right)$$
$$=4\pi\left(1-\tfrac{1}{e}\right) \quad \text{units}^3$$

(b)
$$2x-x^2=kx$$
$$x^2+(k-2)x=0$$
$$x(x+k-2)=0$$
$$\therefore x=0,\ 2-k$$

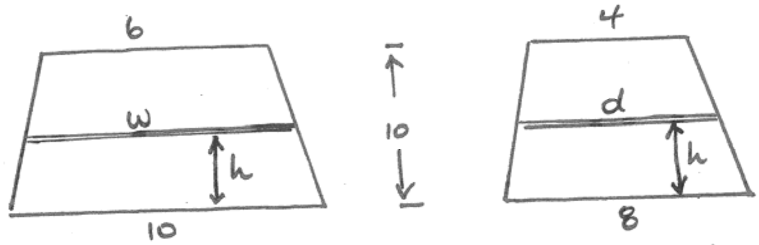


$$\Delta v=\pi\left((x+\Delta x)^2-x^2\right)\cdot(2x-x^2-kx)$$
$$=\pi\left(x^2+2x\,\Delta x+(\Delta x)^2-x^2\right)(2x-x^2-kx)$$
$$=2\pi x(2x-x^2-kx)\,\Delta x$$

(ignoring $(\Delta x)^2$ term) – negligible

$$\therefore V=2\pi\int_0^{2-k}\left((2-k)x^2-x^3\right)dx$$
$$=2\pi\left[\frac{2-k}{3}x^3-\frac{1}{4}x^4\right]_0^{2-k}$$
$$=2\pi\left[\frac{(2-k)^4}{3}-\frac{(2-k)^4}{4}-0\right]$$
$$=\frac{\pi(2-k)^4}{6} \quad \text{units}^3$$

(c)



$w = 10 - kh$

when $h = 10$ $w = 6$

$\therefore\ 6 = 10 - 10k$

$10k = 4$

$k = \frac{2}{5}$

$\therefore\ w = 10 - \frac{2}{5}h$

$d = 8 - ch$

when $h = 10$ $d = 4$

$\therefore\ 4 = 8 - 10c$

$10c = 4$

$c = \frac{2}{5}$

$\therefore\ d = 8 - \frac{2}{5}h$

$$\begin{aligned}
\therefore\quad V &= \int_0^{10} \left(10 - \tfrac{2}{5}h\right)\left(8 - \tfrac{2}{5}h\right) dh \\
&= \int_0^{10} \left(80 - \tfrac{36}{5}h + \tfrac{4}{25}h^2\right) dh \\
&= \left[80h - \tfrac{18}{5}h^2 + \tfrac{4}{75}h^3\right]_0^{10} \\
&= 800 - \tfrac{1800}{5} + \tfrac{4000}{75} - 0 \\
&= \tfrac{1480}{3} \quad \text{units}^3
\end{aligned}$$

Question 6

(a) (i) zeros are $a+bi,\ a-bi,\ a+2bi,\ a-2bi$

Σ roots $4a = 4 \quad \therefore a=1$

Product roots $= (a^2+b^2)(a^2+4b^2)$

$\therefore \quad 10 = (1+b^2)(1+4b^2)$

$\therefore \quad 4b^4+5b^2-9=0$

$(4b^2+9)(b^2-1)=0$

$\therefore \quad b=\pm 1 \qquad (b\in\mathbb{R})$

$\therefore a=1 \quad b=\pm 1$

(ii) $1+i,\ 1-i,\ 1+2i,\ 1-2i$

$\Sigma=2 \quad \alpha\beta=2 \qquad \Sigma=2 \quad \alpha\beta=5$

$\therefore \quad P(x)=(x^2-2x+2)(x^2-2x+5)$

(b)(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2\Sigma\alpha\beta$

$= (-2)^2-2(-3)$

$= 10$

(ii) $\frac{1}{\alpha^2\beta^2}+\frac{1}{\beta^2\gamma^2}+\frac{1}{\gamma^2\alpha^2} = \frac{\alpha^2+\beta^2+\gamma^2}{(\alpha\beta\gamma)^2}$

$= \frac{10}{4^2}$

$= \frac{5}{8}$

(iii) $\left(\frac{2}{x}\right)^3+2\left(\frac{2}{x}\right)^2-3\left(\frac{2}{x}\right)-4=0$

$\frac{8}{x^3}+\frac{8}{x^2}-\frac{6}{x}-4=0$

$8+8x-6x^2-4x^3=0$

ie $\quad 2x^3+3x^2-4x-4=0$

(iv) $\frac{\beta\gamma}{\alpha},\ \frac{\gamma\alpha}{\beta},\ \frac{\alpha\beta}{\gamma} = \frac{\alpha\beta\gamma}{\alpha^2},\ \frac{\alpha\beta\gamma}{\beta^2},\ \frac{\alpha\beta\gamma}{\gamma^2}$

ie $\frac{4}{\alpha^2},\ \frac{4}{\beta^2},\ \frac{4}{\gamma^2}$

$\left(\sqrt{\frac{4}{x}}\right)^3+2\left(\sqrt{\frac{4}{x}}\right)^2-3\left(\sqrt{\frac{4}{x}}\right)-4=0$

$\frac{8}{x\sqrt{x}}+\frac{8}{x}-\frac{6}{\sqrt{x}}-4=0$

$8+8\sqrt{x}-6x-4x\sqrt{x}=0$

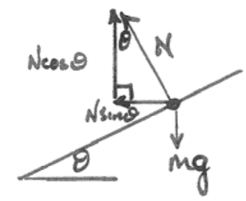
$(8-4x)\sqrt{x} = 6x-8$

$(8-4x)^2x = (6x-8)^2$

$$(64 - 64x + 16x^2)x = 36x^2 - 96x + 64$$
$$64x - 64x^2 + 16x^3 = 36x^2 - 96x + 64$$
$$16x^3 - 100x^2 + 160x - 64 = 0$$
$$4x^3 - 25x^2 + 40x - 16 = 0 .$$

Question 7

(a) $r = 4000$ $\quad 160\text{ km/h} = \frac{400}{9}\text{ ms}^{-1}$



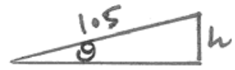
$$N\cos\theta = mg$$
$$N\sin\theta = mr\omega^2$$
$$\tan\theta = \frac{mr\omega^2}{mg}$$

$V = r\omega$

$\frac{400}{9} = 4000\,\omega$

$\therefore \omega = \frac{1}{90}$

$$\therefore \tan\theta = \frac{4000\left(\frac{1}{90}\right)^2}{9.8}$$
$$\therefore \theta = 2.8847\ldots^\circ$$



$$\sin\theta = \frac{h}{1.5}$$
$$\therefore h = 1.5\sin\theta$$
$$= 0.07549\ldots$$

$\therefore$ Approx 7.55 cm

(b)(i) $a = g - kv^2$

For terminal velocity $a = 0$

$$\therefore g - kv^2 = 0$$
$$v^2 = \frac{g}{k}$$
$$\therefore v = \sqrt{\frac{g}{k}}$$

(ii)(α) $a = -g - kv^2$

$$v\frac{dv}{dx} = -g - kv^2$$
$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$
$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$
$$= -\frac{1}{2k}\cdot\frac{2kv}{g + kv^2}$$
$$x = -\frac{1}{2k}\ln(g + kv^2) + c$$

when $x = 0$ $\quad v = u$

$$\therefore 0 = -\frac{1}{2k}\ln(g + ku^2) + c$$
$$\therefore x = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g + kv^2}\right)$$

Max height when $v = 0$

$$\therefore \text{Max height} = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)$$

(β) $a = -g - kv^2$

$$\frac{dv}{dt} = -g - kv^2$$

$$\frac{dt}{dv} = -\frac{1}{g+kv^2}$$

$$= -\frac{1}{\sqrt{g}} \cdot \frac{\sqrt{g}}{g + (\sqrt{k}\,v)^2}$$

$$t = -\frac{1}{\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot \tan^{-1}\sqrt{\frac{k}{g}}\,v + c$$

$$= -\frac{1}{\sqrt{gk}} \tan^{-1}\sqrt{\frac{k}{g}}\,v + c$$

when $t = 0 \quad v = u$

$$\therefore 0 = -\frac{1}{\sqrt{gk}} \tan^{-1}\sqrt{\frac{k}{g}}\,u + c$$

$$\therefore t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\sqrt{\frac{k}{g}}\,u - \tan^{-1}\sqrt{\frac{k}{g}}\,v\right)$$

when $v = 0 \quad t = \frac{1}{\sqrt{gk}} \tan^{-1}\sqrt{\frac{k}{g}}\,u$

Question 8

(a) (i) $\angle PLA + \angle PMA = 180°$
$\therefore$ P,M,A,L are concyclic (opp $\angle$'s supp)

(ii) $\angle PMB = \angle PNB = 90°$
$\therefore$ P,B,N,M are concyclic ($\angle$'s standing on same chord equal)

(iii) Join LM, MN and PB.
Let $\angle AML = x$
$\therefore \angle APL = x$ ($\angle$'s standing on same arc in circle PMAL)
$\therefore \angle PAL = 90 - x$ ($\angle$ sum ΔPLA)
$\therefore \angle PBC = 90 - x$ (opp $\angle$'s cyclic quad PACB)
$\therefore \angle BPN = x$ ($\angle$ sum ΔBPN)
$\therefore \angle BMN = x$ ($\angle$'s standing on same arc in circle PBNM)
$\therefore \angle AML = \angle BMN$
But BMA is straight
$\therefore$ LMN are collinear (vert opp $\angle$'s equal)

(b) (i) $J_n = \int_0^1 x^n e^{-x}\,dx$
$\therefore J_0 = \int_0^1 x^0 e^{-x}\,dx$
$= \left[-e^{-x}\right]_0^1$
$= -e^{-1} - -e^0$
$= 1 - \frac{1}{e}$

(ii) $J_n = \int_0^1 x^n e^{-x}\,dx$
$= \left[-e^{-x} \cdot x^n\right]_0^1 - \int_0^1 -e^{-x} \cdot nx^{n-1}\,dx$
$= (-e^{-1} \cdot 1^n - -e^0 \cdot 0) + n\int_0^1 x^{n-1} e^{-x}\,dx$
$= n\,J_{n-1} - \frac{1}{e}$

(iii) RTP $J_n = n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}$

Prove true for $n=0$ $\quad J_0 = 0! - \frac{0!}{e} \sum_{k=0}^{0} \frac{1}{0!}$

$= 1 - \frac{1}{e}$

Assume true for n

ie $J_n = n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}$

Prove true for $n+1$

$J_{n+1} = (n+1) J_n - \frac{1}{e}$ (from (ii))

$= (n+1)\left[n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}\right] - \frac{1}{e}$ (using assumption)

$= (n+1)! - \frac{(n+1)!}{e} \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{e}$

$= (n+1)! - \frac{(n+1)!}{e} \sum_{k=0}^{n} \frac{1}{k!} - \frac{(n+1)!}{e} \cdot \frac{1}{(n+1)!}$

$\therefore J_{n+1} = (n+1)! - \frac{(n+1)!}{e} \sum_{k=0}^{n+1} \frac{1}{k!}$

$\therefore$ True for $n+1$

Since true for $n=0$ then true for $n=0+1=1$, $1+1=2, \ldots$ $\therefore$ true for all integers $n \geq 0$.

(iv) $\int_0^1 x^n e^{-x}\,dx$ represents the area under $y = x^n e^{-x}$ between $x=0$ and $x=1$.

As $n \to \infty$ $\quad x^n \to 0$ for $0 < x < 1$

$\therefore \int_0^1 x^n e^{-x} = J_n \to 0$.

(v) $\lim_{n\to\infty} J_n = \lim_{n\to\infty} \left(n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}\right)$

$\therefore 0 = \lim_{n\to\infty} \left(n! - \frac{n!}{e} \sum_{k=0}^{n} \frac{1}{k!}\right)$

$\therefore \lim_{n\to\infty} \frac{1}{e} \sum_{k=0}^{n} \frac{1}{k!} = 1$

ie $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = e$